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THE PROPERTIES OF A CLASS OF INVARIANT RELATIONS OF THE GENERALIZED EQUATIONS OF DYNAMICS†

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The existence of invariant relations is investigated for the generalized Euler-Poisson equations of the dynamics of a rigid body with a fixed point, for which integrals of energy, angular momentum and the geometric integral exist. Different formulations are considered, when the potentials of the applied forces are known and when they are determined according to the form of the invariant relations from the expressions for the first integrals. Earlier results [1] are extended to the case of one invariant relation of arbitrary structure, and the conditions for invariant relation of layer zero (in P. V. Kharlamov's terminology [2]) to exist are considered. © 2001 Elsevier Science Ltd. All rights reserved.

The method of invariant relations (IRs) [3, 4], as developed up to an algorithm by P. V. Kharlamov [2], enables one to investigate IRs by either of two approaches. The first presupposes the use of the differential equations of motion only. The second is based on the use of integrals. Using the second approach, existence conditions have been found [1] for three linear IRs of the Grioli–M. P. Kharlamov differential equations [5, 6].

In this paper an example is presented which shows that, for given force and gyroscopic functions, the existence conditions for three linear IRs of the Grioli–M. P. Kharlamov equations may not be the same as the conditions of [1]. One invariant condition is considered and an additional restriction is obtained, under which the results of [1] can be generalized. New classes of force and gyroscopic functions are established for the Grioli–M. P. Kharlamov equations, guaranteeing the existence of a single linear IR of layer zero.

1. FORMULATION OF THE PROBLEM

Consider the problem of the motion of a rigid body in a field of potential and gyroscopic forces, on the assumption that, as in the classical case, the differential equations of motion admit of three first integrals. In vector notation, we then have the following equations [5]

$$\dot{x} = x \times ax + \mu(\nu, ax)(\nu \times ax) + \frac{\partial L}{\partial \nu} \times ax + \frac{\partial U}{\partial \nu} \times \nu \tag{1.1}$$

$$\dot{\nu} = \nu \times ax \tag{1.2}$$

$$\left(\frac{\partial L}{\partial \nu} = \left(\frac{\partial L}{\partial \nu_1}, \frac{\partial L}{\partial \nu_2}, \frac{\partial L}{\partial \nu_3}\right), \frac{\partial U}{\partial \nu} = \left(\frac{\partial U}{\partial \nu_1}, \frac{\partial U}{\partial \nu_2}, \frac{\partial U}{\partial \nu_3}\right)\right)$$

where x is the angular momentum vector of the body, ν is a unit vector along the axis of symmetry of the force field, $L(\nu_1, \nu_2, \nu_3)$, $U(\nu_1, \nu_2, \nu_3)$ are scalar functions of the components of the vector
ν, $\mu(\nu, ax)$ is a scalar function of the components of the vectors ν and $\omega = ax$, $\omega = (\omega_1, \omega_2, \omega_3)$ with

$$\omega_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \quad \omega_2 = a_{12}x_1 + a_{22}x_2 + a_{23}x_3$$
$$\omega_3 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \tag{1.3}$$

x_1, x_2, x_3 are the components of the vector x, and a_{ij} are the components of the tensor a.

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In classical problems, $\mu \equiv 0$ and $L(\nu_1, \nu_2, \nu_3)$ and $U(\nu_1, \nu_2, \nu_3)$ are polynomials in ν_i of up to the second order. If it is assumed that the function μ depends only on ν_i ($i = 1, 2, 3$), we obtain M. P. Kharlamov's equation [6]. This case is also considered below.

Equations (1.1) and (1.2) have three first integrals:

$$ax \cdot x - 2U(\nu_1, \nu_2, \nu_3) = 2E, \quad x \cdot \nu + L(\nu_1, \nu_2, \nu_3) = k, \quad \nu \cdot \nu = 1 \tag{1.4}$$

where E and k are arbitrary constants.

Suppose one is faced with the problem of the conditions for the existence of an IR

$$f(x_1, x_2, x_3, \nu_1, \nu_2, \nu_3) = 0 \tag{1.5}$$

for Eqs (1.1) and (1.2).

In the first approach to using the IR method [2], one evaluates the derivatives of the function on the left of Eq. (1.5) along trajectories of Eqs (1.1) and (1.2). If the manifold defined by Eq. (1.5) and the conditions obtained by equating these derivatives to zero is not empty, it is called an invariant manifold of system (1.1), (1.2), and Eq. (1.5) is called an IR [2]. A special case, considered by Poincaré [4] and Levi–Civita [5], is obtained when the derivative of the function on the left of Eq. (1.5) along trajectories of Eqs (1.1) and (1.2) is equal to zero on relation (1.5). In the general case, in order to establish that (1.5) is an IR, one has to investigate the functional dependence of (1.5) and the derivatives evaluated as stated above. If the sequence consisting of (1.5) and the derivatives contains l functionally independent terms, then (1.5) is called an IR of layer $l - 1$ [2].

The characteristic aspects of the application of the IR method are demonstrated by the following example. Suppose we are given a system of three differential equations

$$\dot{x} = x, \quad \dot{y} = x^2 + z^2 - b^2, \quad \dot{z} = -(2x^2 + z^2 - b^2)z^{-1}$$

where b is a constant parameter. For this system, the relation $x = 0$ along trajectories of $\dot{x} = x = 0$ is an IR of layer zero, and the relation $y = 0$ along trajectories of the equalities

$$\dot{y} = x^2 + z^2 - b^2 = 0, \quad \ddot{y} = x^2 + z^2 - b^2 = 0$$

is an IR of layer one. In the first case, the derivative of the IR vanishes identically on the given IR; in the second case, the second derivative of the IR vanishes on the given IR and the set defined by the first derivative.

P. V. Kharlamov's second approach [2], based on using the first integrals of the differential equations of motion, consists in adding the integrals (1.4) to relation (1.5) and the conditions equating the derivatives of the function on the left of (1.5) along trajectories of Eqs (1.1) and (1.2) to zero. Investigating the functional dependence of the terms of the sequence thus constructed, one arrives at the conditions for IR (1.5) to exist. The definition of the layer of the IR remains unchanged [2].

Thus, when investigating IRs, it is always necessary to specify the layer of the IR, or to define the properties of the IR with respect to system (1.1)–(1.4).

2. THREE IRs. ORESHKINA'S RESULTS [1]

In [1], three IRs of Eqs (1.1) and (1.2) with $\mu = \mu(\nu_1, \nu_2, \nu_3)$ were considered

$$x_i = g_i(\nu_1, \nu_2, \nu_3), \quad i = 1, 2, 3 \tag{2.1}$$

Using the integrals (1.4), the functions $L = L(\nu_1, \nu_2, \nu_3)$, $U = U(\nu_1, \nu_2, \nu_3)$ were determined on the basis of Eqs (1.3). Substitution of these functions into Eqs (1.1) and (1.2) gives three relations:

$$(\omega_3\nu_2 - \omega_2\nu_3)\left[\mu(\nu_1, \nu_2, \nu_3) - \frac{\partial g_1}{\partial \nu_1} - \frac{\partial g_2}{\partial \nu_2} - \frac{\partial g_3}{\partial \nu_3}\right] = 0 \ (123) \tag{2.2}$$

The symbol (123) means that the unwritten equations are obtained from (2.2) by a cyclic permutation of the numbers 1, 2, 3. Since the equality $\omega_3\nu_2 - \omega_2\nu_3 = 0$ (123) along trajectories of Eq. (1.2) leads to stationary solutions of Eqs (1.1) and (1.2), it follows from (2.2) that [1]

$$\mu(\nu_1,\nu_2,\nu_3)=\frac{\partial g_1}{\partial\nu_1}+\frac{\partial g_2}{\partial\nu_2}+\frac{\partial g_3}{\partial\nu_3} \tag{2.3}$$

Since only the first derivatives of (2.2) have been considered, this means that the functions $L = L(\nu_1, \nu_2, \nu_3)$, $U = U(\nu_1, \nu_2, \nu_3)$, $\mu = \mu(\nu_1, \nu_2, \nu_3)$ defined in [1] were such that Eqs (1.1) and (1.2) admit of three IRs (2.1) of layer zero. It follows from this method that relation (2.3) must hold for all values of ν_1, ν_2, ν_3 in the space R^3.

Consider the following example, which shows that the approach described previously in [1] is inapplicable for specific functions $L = L(\nu_1, \nu_2, \nu_3)$, $U = U(\nu_1, \nu_2, \nu_3)$.

Suppose the components of the gyration tensor a_{ij} are $a_{ij} = 0$ $(i \neq j)$, $a_{ii} = a_i$, and $L(\nu_1, \nu_2, \nu_3)$, $U(\nu_1, \nu_2, \nu_3)$ are the following quadratic forms

$$L(\nu_1,\nu_2,\nu_3)=\frac{b_2}{a_1}(a_1\nu_2^2-\alpha_0 a_2\nu_1^2),\quad U(\nu_1,\nu_2,\nu_3)=\frac{a_2b_2^2}{2a_1}(a_1\nu_2^2+\alpha_0^2a_2\nu_1^2) \tag{2.4}$$

where b_2 and α_0 are parameters.

We specify three IRs of Eqs (1.1) and (1.2)

$$x_1=a_2b_2(e_1\nu_1^2+e_0)\nu_1D^{-1},\quad x_2=b_2(f_1\nu_1^2+f_0)\nu_2D^{-1} \tag{2.5}$$
$$x_3=2a_2b_2(g_1'\nu_1^2+g_0)\nu_3D^{-1}$$

where

$$\nu_2^2(\nu_1)=\alpha_0\nu_1^2+\beta_0,\quad \nu_3^2(\nu_1)=1-\beta_0-(1+\alpha_0)\nu_1^2$$
$$\dot\nu_1=-a_2b_2\sqrt{\nu_2^2(\nu_1)\nu_3^2(\nu_2)} \tag{2.6}$$

We have used the following notation in (2.5) and (2.6)

$$D=d_1\nu_1^2+d_0,\quad d_1=\alpha_0a_1(a_2-a_3)+a_2(a_1-a_3),\quad d_0=a_1[(a_2-a_3)\beta_0-a_2]$$
$$e_1=\alpha_0[\alpha_0a_1(a_3-a_2)+2a_1a_3-a_2(a_1+a_3)] \tag{2.7}$$
$$e_0=a_1[2a_3\beta_0+\alpha_0a_2+\alpha_0\beta_0(a_3-a_2)]$$
$$f_1=\alpha_0(a_1a_3+a_1a_2-2a_2a_3)-a_2(a_3-a_1),\quad f_0=a_1[(a_2+a_3)\beta_0-a_2]$$
$$g_1'=\alpha_0(a_1-a_2),\quad g_0=a_1\beta_0$$

where β_0 is an arbitrary parameter.

It can be shown that relations (2.4)–(2.7) define a solution of Eqs (1.1) and (1.2) characterized by the three IRs (2.5), provided that the function $\mu(\nu_1, \nu_2, \nu_3) = \mu(\nu_1, \nu_2(\nu_1), \nu_3(\nu_1)) = \mu^*(\nu_1)$ takes the following value (the prime denotes differentiation with respect to ν_1)

$$\mu^*(\nu_1)=a_2b_2a_1^{-1}[(e_1\nu_1^2+e_0)\nu_1D^{-1}]'+F(\nu_1) \tag{2.8}$$
$$F(\nu_1)=2b_2(a_3-a_2)(f_1\nu_1^2+f_0)(g_1'\nu_1^2+g_0)D^{-2}+4b_2a_3(g_1'\nu_1^2+g_0)D^{-1}+b_2$$

If one uses Oreshkina's method [1], the value obtained from (2.3), (2.5) and (2.7) for the function (2.3) on solution (2.6) differs from (2.8) in that now

$$F(\nu_1)=b_2([f_1+2a_2g_1)\nu_1+(f_0+2a_2g_0)]D^{-1}$$

As will be explained below, the reason for this difference is that IR (2.1) and IR (2.5) belong to different layers, while the earlier approach in [1] makes no reference to the layer of the IR.

3. INVESTIGATION OF A SINGLE IR USING INTEGRALS

Suppose we are given a single IR

$$x_1 = g_1(\nu_1, \nu_2, \nu_3) \tag{3.1}$$

Let us assume that $L = L(\nu_1, \nu_2, \nu_3)$, $U = U(\nu_1, \nu_2, \nu_3)$ are given differential functions of the variables ν_1, ν_2 and ν_3. We will also assume that (3.1) is an IR of layer zero in P. V. Kharlamov's terminology [2]. To investigate IR (3.1), we will use the first integrals (1.4). We substitute expression (3.1) into (1.4) and choose a system of coordinates so that $a_{23} = 0$. We obtain the following relations

$$a_{22}x_2^2 + a_{33}x_3^2 + 2(a_{12}x_2 + a_{13}x_3)g_1(\nu_1, \nu_2, \nu_3) = \varphi_1(\nu_1, \nu_2, \nu_3) \tag{3.2}$$

$$x_2\nu_2 + x_3\nu_3 = \varphi_2(\nu_1, \nu_2, \nu_3) \tag{3.3}$$

where

$$\begin{aligned} &\varphi_1(\nu_1, \nu_2, \nu_3) = 2E - a_{11}g_1^2(\nu_1, \nu_2, \nu_3) + 2U(\nu_1, \nu_2, \nu_3) \\ &\varphi_2(\nu_1, \nu_2, \nu_3) = K - \nu_1 g_1(\nu_1, \nu_2, \nu_3) - L(\nu_1, \nu_2, \nu_3) \end{aligned} \tag{3.4}$$

If follows from (3.2) and (3.3) that

$$\begin{aligned} x_2 = g_2(\nu_1, \nu_2, \nu_3) = \Delta_1^{-1}(a_{33}\nu_2\varphi_2 + g_1\nu_3\Delta_2 + \nu_3\sqrt{\Delta}) \\ x_3 = g_3(\nu_1, \nu_2, \nu_3) = \Delta_1^{-1}(a_{22}\nu_3\varphi_2 - g_1\nu_2\Delta_2 - \nu_2\sqrt{\Delta}) \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} &\Delta_1 = a_{22}\nu_3^2 + a_{33}\nu_2^2, \quad \Delta_2 = a_{13}\nu_2 - a_{12}\nu_3 \\ &\Delta = g_1^2\Delta_2^2 - 2g_1(a_{12}a_{33}\nu_2 + a_{22}a_{13}\nu_3)\varphi_2 - a_{22}a_{33}\varphi_2^2 + \Delta_1\varphi_1 \end{aligned} \tag{3.6}$$

and φ_1 and φ_2 are given by formulae (3.4). Since (3.1) is an IR of layer zero, the equation $\dot{x}_1 - \dot{g}_1(\nu_1, \nu_2, \nu_3) = 0$ is satisfied, or, because of relations (1.1)–(1.3),

$$\begin{aligned} &\omega_3\left(x_2 + \mu(\nu_1, \nu_2, \nu_3)\nu_2 + \frac{\partial L}{\partial \nu_2}\right) - \omega_2\left(x_3 + \mu(\nu_1, \nu_2, \nu_3)\nu_3 + \frac{\partial L}{\partial \nu_3}\right) + \nu_3\frac{\partial U}{\partial \nu_2} - \\ &-\nu_2\frac{\partial U}{\partial \nu_3} - (\omega_3\nu_2 - \omega_2\nu_3)\frac{\partial g_1}{\partial \nu_1} - (\omega_1\nu_3 - \omega_3\nu_1)\frac{\partial g_1}{\partial \nu_2} - (\omega_2\nu_1 - \omega_1\nu_2)\frac{\partial g_1}{\partial \nu_3} = 0 \end{aligned} \tag{3.7}$$

with ω_i as in (1.3), in which we must set $a_{23} = 0$ and $x_1 = g_1(\nu_1, \nu_2, \nu_3)$.

Now substitute expressions (1.3) and (3.5) into Eq. (3.7)

$$\begin{aligned} &\Delta_1^2\left[\left(\frac{\partial g_1}{\partial \nu_1} - \mu(\nu_1, \nu_2, \nu_3)\right)\sqrt{\Delta} + \left(a_{13}\frac{\partial L}{\partial \nu_2} - a_{12}\frac{\partial L}{\partial \nu_3}\right)g_1 + \Delta_3 g_1\frac{\partial g_1}{\partial \nu_2} + \right. \\ &\left. +\Delta_4 g_1\frac{\partial g_1}{\partial \nu_3} + \nu_3\frac{\partial U}{\partial \nu_2} - \nu_2\frac{\partial U}{\partial \nu_3}\right] + (a_{33} - a_{22})u_2u_3 + \\ &+\Delta_1\left[u_2\left(a_{13}g_1 - a_{12}\nu_3\frac{\partial g_1}{\partial \nu_2} - \Delta_5\frac{\partial g_1}{\partial \nu_3} - a_{22}\frac{\partial L}{\partial \nu_3}\right) - \right. \\ &\left. -u_3\left(a_{12}g_1 + \Delta_6\frac{\partial g_1}{\partial \nu_2} - a_{13}\nu_2\frac{\partial g_1}{\partial \nu_3} - a_{33}\frac{\partial L}{\partial \nu_2}\right)\right] = 0 \end{aligned} \tag{3.8}$$

where

$$\Delta_3 = a_{13}v_1 - a_{11}v_3,\quad \Delta_4 = a_{11}v_2 - a_{12}v_1,\quad \Delta_5 = a_{22}v_1 - a_{12}v_2,\quad \Delta_6 = a_{13}v_3 - a_{33}v_1$$

$$u_2 = a_{33}v_2\varphi_2 + g_1v_3\Delta_2 + v_3\sqrt{\Delta},\quad u_3 = a_{22}v_3\varphi_2 - g_1v_2\Delta_2 - v_2\sqrt{\Delta}$$

It can be shown that if $g_1(v_1, v_2, v_3)$ satisfies the equation $\Delta = 0$, then Eq. (3.8) is satisfied. Indeed, it will suffice to express Eq. (3.8), subject to the condition $\Delta = 0$, in the form

$$\frac{1}{\theta(v_1,v_2,v_3)}\frac{d\Delta}{dt}\Big|_{\Delta=0} = 0$$

where

$$\theta(v_1,v_2,v_3) = \Delta_1^{-1}\{g_1[(a_{11}a_{33} - a_{13}^2)v_2^2 + 2a_{12}a_{13}v_2v_3 + (a_{11}a_{22} - a_{12}^2)v_3^2 - -(a_{12}a_{33}v_2 + a_{13}a_{22}v_3)v_1] - \varphi_2(a_{22}a_{33}v_1 - a_{12}a_{33}v_2 - a_{13}a_{22}v_3)\} \neq 0$$

because of the fact that v_1, v_2 and v_3 cannot simultaneously be constants.

The form of Eq. (3.8) shows that, for the prescribed functions $L = L(v_1, v_2, v_3)$, $U = U(v_1, v_2, v_3)$, the function $\mu(v_1, v_2, v_3)$ may be expressed, provided that $\Delta \neq 0$, in terms of $g_1(v_1, v_2, v_3)$, $L(v_1, v_2, v_3)$, $U(v_1, v_2, v_3)$ and their derivatives. This fact distinguishes the case of a single IR from the case of three IRs [1], for which the function $\mu(v_1, v_2, v_3)$ is defined by (2.3).

Nevertheless, even in the case of a single IR, property (2.3), which was masked by the use of Eq. (3.8), will hold. If we assume that $\Delta \neq 0$, then Eqs (3.5) and (3.6) may be used to show that (2.3) follows from (3.8). This implies that in the investigation of a single IR of layer zero, a necessary condition for Eqs (1.1) and (1.2) to have such an IR is

$$\left(\mu(v_1,v_2,v_3) - \frac{\partial g_1}{\partial v_1} - \frac{\partial g_2}{\partial v_2} - \frac{\partial g_3}{\partial v_3}\right)\Delta(v_1,v_2,v_3) = 0 \quad (3.9)$$

where g_2 and g_3 are defined by (3.5). Thus, property (2.3) is a necessary condition for the existence of a single IR (3.1), but only provided that $\Delta \neq 0$. Here the IR is investigated with the help of first integrals.

The singular case $\Delta = 0$, as already observed, yields no conditions imposed on the function $\mu(v_1, v_2, v_3)$.

Within the context of this version, it can be shown that the equations for $\dot{x}_2$, $\dot{x}_3$ in (1.1) cannot be replaced by integrals (3.2) and (3.3), so that the case $\Delta = 0$ requires further investigation. However, if one formally proceeds using the IR method of [2], which uses first integrals, one must consider Eq. (3.9) as a necessary condition. It is in this sense that the method of [2] is an improvement over Oreshkina's results [1], as applied to the consideration of a single IR. It is obvious that (2.3) is a sufficient existence condition both in the study of three IRs [1] and in the investigation of a single IR.

Remark. Condition (2.3) plays different roles in the consideration of the direct and inverse problems for Eqs (1.1) and (1.2). If three IRs are given, then, following the earlier approach [1], a complete solution of the problem (albeit a fairly trivial one) is in reach. When one is investigating a single IR and the functions $L(v_1, v_2, v_3)$, $U(v_1, v_2, v_3)$ are given, then the definition of $\mu(v_1, v_2, v_3)$ or $g_1(v_1, v_2, v_3)$ in the case $\Delta \neq 0$ requires the use of Eq. (3.8), but Eq. (2.3) is only a compact notation for (3.8), given that equalities (3.5) are true. In that case, however, condition (2.3) reflects a property of differential equations (1.1) and (1.2) that admit of IR (3.1) and its corollaries – IRs (3.5).

The case in which Eqs (1.1) and (1.2) describe classical problems, that is, when $\mu \equiv 0$, is of particular interest. Condition (2.3) then becomes

$$\frac{\partial g_1}{\partial v_1} + \frac{\partial g_2}{\partial v_2} + \frac{\partial g_3}{\partial v_3} = 0 \quad (3.10)$$

which may be regarded as a condition for the existence of a single IR of layer zero for system (1.1), (1.2) when first integrals are used. Condition (3.10) will then hold for all v_1, v_2 and v_3 in R^3 and, if it is used in the energy and area integrals of (1.4), the relation $v_1^2 + v_2^2 + v_3^2 = 1$ cannot be used.

To demonstrate this, let us consider Eqs (1.1) and (1.2) subject to the conditions

$$\mu(\nu_1,\nu_2,\nu_3)\equiv 0,\quad L(\nu_1,\nu_2,\nu_3)\equiv 0,\quad U(\nu_1,\nu_2,\nu_3)=\Gamma(e\cdot\nu),\quad \Gamma=\text{const}$$
$$e=(1,0,0),\quad a_{11}=a,\quad a_{22}=a_{33}=a_1,\quad a_{12}=b_1,\quad a_{13}=a_{23}=0$$

Then the equations of motion are

$$\begin{aligned}
&\dot{x}_1=-b_1x_1x_3,\quad \dot{x}_2=(a-a_1)x_1x_3+b_1x_2x_3-\Gamma\nu_3\\
&\dot{x}_3=-(a-a_1)x_1x_2+b_1(x_1^2-x_2^2)+\Gamma\nu_2\\
&\dot{\nu}_1=a_1x_3\nu_2-(a_1x_2+b_1x_1)\nu_3,\quad \dot{\nu}_2=(ax_1+b_1x_2)\nu_3-a_1x_3\nu_1\\
&\dot{\nu}_3=(a_1x_2+b_1x_1)\nu_1-(ax_1+b_1x_2)\nu_2
\end{aligned}\tag{3.11}$$

The integrals of Eqs (3.11) are

$$\begin{aligned}
&\nu_1^2+\nu_2^2+\nu_3^2=1\\
&ax_1^2+a_1(x_2^2+x_3^2)+2b_1x_1x_2-2\Gamma\nu_1=2E,\quad x_1\nu_1+x_2\nu_2+x_3\nu_3=k
\end{aligned}\tag{3.12}$$

Equations (3.1) admit of the well-known Hess IR $x_1 = 0$ of layer zero.
Let us find x_2 and x_3 from (3.12), assuming $x_1 = 0$:

$$\begin{aligned}
&x_2=N_1^{-1}(k\sqrt{a_1}\nu_2+\nu_3\sqrt{N}),\quad x_3=N_1^{-1}(k\sqrt{a_1}\nu_3-\nu_2\sqrt{N})\\
&N_1=\sqrt{a_1}(\nu_2^2+\nu_3^2),\quad N=2(\nu_2^2+\nu_3^2)(\Gamma\nu_1+E)-k^2a_1
\end{aligned}\tag{3.13}$$

Using the equalities $x_1 = 0$ and (3.13), one readily verifies the truth of equality (3.10). However, if one puts $\nu_2^2 + \nu_3^2 = 1 - \nu_1^2$ in (3.13) and substitutes the results into (3.10), the result is an expression which is non-zero.

4. INVESTIGATION OF A SINGLE IR WITHOUT THE USE OF INTEGRALS

Let (3.1) be an IR of layer zero without the use of integrals. The functions $L = L(\nu_1, \nu_2, \nu_3)$, $U = U(\nu_1, \nu_2, \nu_3)$ have to be determined. Since a single IR is given, these functions cannot be determined from the integrals (1.4). We substitute (3.1) into (1.3) and then substitute the ω_i thus found and (3.1) into Eq. (3.7). The requirement that the resulting equality must hold for any x_2 and x_3 leads to the following conditions

$$a_{22}=a_{33},\quad a_{12}=0,\quad a_{23}=0\tag{4.1}$$

$$a_{22}\left(\nu_3\mu(\nu_1,\nu_2,\nu_3)+\frac{\partial L}{\partial\nu_3}\right)=a_{13}g_1+a_{22}\left(\nu_3\frac{\partial g_1}{\partial\nu_1}-\nu_1\frac{\partial g_1}{\partial\nu_3}\right)\tag{4.2}$$

$$a_{22}\left(\nu_2\mu(\nu_1,\nu_2,\nu_3)+\frac{\partial L}{\partial\nu_2}\right)=a_{13}\left(\nu_3\frac{\partial g_1}{\partial\nu_2}-\nu_2\frac{\partial g_1}{\partial\nu_3}\right)+a_{22}\left(\nu_2\frac{\partial g_1}{\partial\nu_1}-\nu_1\frac{\partial g_1}{\partial\nu_2}\right)\tag{4.3}$$

$$a_{22}\left(\nu_3\frac{\partial U}{\partial\nu_2}-\nu_2\frac{\partial U}{\partial\nu_3}\right)+(a_{11}a_{22}-a_{13}^2)\left(\nu_2\frac{\partial g_1}{\partial\nu_3}-\nu_3\frac{\partial g_1}{\partial\nu_2}\right)g_1=0\tag{4.4}$$

Eliminating the function $\mu(\nu_1, \nu_2, \nu_3)$ from equalities (4.2) and (4.3), we have

$$a_{22}\left(\nu_3\frac{\partial L}{\partial\nu_2}-\nu_2\frac{\partial L}{\partial\nu_3}\right)+a_{13}\nu_2g_1+(a_{22}\nu_1-a_{13}\nu_3)\left(\nu_3\frac{\partial g_1}{\partial\nu_2}-\nu_2\frac{\partial g_1}{\partial\nu_3}\right)=0\tag{4.5}$$

To solve differential equations (4.4) and (4.5), we introduce new functions

$$\psi_1(\nu_1,\nu_2,\nu_3)=\frac{1}{2}(a_{11}a_{22}-a_{13}^2)g_1^2(\nu_1,\nu_2,\nu_3)-a_{22}U(\nu_1,\nu_2,\nu_3) \tag{4.6}$$

$$\psi_2(\nu_1,\nu_2,\nu_3)=a_{22}L(\nu_1,\nu_2\nu_3)+(a_{22}\nu_1-a_{13}\nu_3)g_1(\nu_1,\nu_2,\nu_3)$$

Applying the general theory of integration of partial differential equations, we deduce from (4.4), (4.6) and (4.2) that

$$U(\nu_1,\nu_2,\nu_3)=a_{22}^{-1}\left[\frac{1}{2}(a_{11}a_{22}-a_{13}^2)g_1^2(\nu_1,\nu_2,\nu_3)+\varphi(\nu_1,\nu_2^2+\nu_3^2)\right] \tag{4.7}$$

$$L(\nu_1,\nu_2,\nu_3)=a_{22}^{-1}\left\lfloor(a_{13}\nu_3-a_{22}\nu_1)g_1(\nu_1,\nu_2,\nu_3)-\Phi(\nu_1,\nu_2^2+\nu_3^2)\right\rfloor \tag{4.8}$$

$$\mu(\nu_1,\nu_2,\nu_3)=a_{22}^{-1}\left(-2\frac{\partial\Phi}{\partial(\nu_1^2+\nu_2^2)}+a_{22}\frac{\partial g_1}{\partial\nu_1}-a_{13}\frac{\partial g_1}{\partial\nu_3}\right) \tag{4.9}$$

where φ and Φ are arbitrary functions, which depend on ν_1 and $\nu_2^2+\nu_3^2$.

Conditions (4.1) are Hess's conditions for the case in which the centre of mass of the body lies on the first coordinate axis Ox_1 (where O is the fixed point). In the case under consideration, since $x_1=g_1(\nu_1,\nu_2,\nu_3)$, this coordinate axis is characterized by the property that the projection of the angular momentum vector onto it is a given function of the components ν_1, ν_2 and ν_3. It is interesting to verify condition (2.3). Having functions (4.7) and (4.8), one can now determine $x_2=g_2(\nu_1,\nu_2,\nu_3)$ and $x_3=g_3(\nu_1,\nu_2,\nu_3)$ from (3.5); then, identical transformations of (4.9), based on (3.5), yield relation (2.3). Thus, this remarkable property also holds for an IR (3.1) of layer zero without the use of integrals.

This result, incidentally, is not trivial, since cases are known in rigid body dynamics in which IRs with different properties (different layers) yield quite different results [7]. For example, in this case, unlike the results of Section 3, the singular case $\Delta=0$ does not arise.

Relations (4.7) and (4.8) are of interest because they enable one to obtain a solution of the problem of investigating a single IR and to construct functions (4.7) and (4.8) using a quite different principle compared with the analogous functions in the earlier method [1]. A mechanical realization of these relations may be established in the class of polynomials in the variables ν_1, ν_2 and ν_3.

5. A SINGLE IR OF GENERAL FORM

We will now generalize the results obtained in Section 3 to the case of an arbitrary IR (1.5), satisfying the condition

$$\left(\frac{\partial f}{\partial x_1}\right)^2+\left(\frac{\partial f}{\partial x_2}\right)^2+\left(\frac{\partial f}{\partial x_3}\right)^2\neq 0$$

Without loss of generality, we can determine

$$x_1=\psi(x_2,x_3,\nu_1,\nu_2,\nu_3) \tag{5.1}$$

from IR (1.5).

Substituting this expression into integrals (1.4), we obtain two equations for x_2 and x_3 as functions of the variables ν_1, ν_2 and ν_3. We will assume that (5.1) is an IR of layer zero in P. V. Kharlamov's terminology [2], that is, the derivative of the function $x_1-\psi(x_2,x_3,\nu_1,\nu_2,\nu_3)$ along trajectories of Eqs (1.1) and (1.2) vanishes on (1.5) at the values found from the integrals:

$$x_2=\theta_2(\nu_1,\nu_2,\nu_3),\quad x_3=\theta_3(\nu_1,\nu_2,\nu_3)$$

Substituting x_2 and x_3 into IR (5.1), we obtain

$$x_1=\psi(\theta_2(\nu_1,\nu_2,\nu_3),\theta_3(\nu_1,\nu_2,\nu_3),\nu_1,\nu_2,\nu_3)=\theta_1(\nu_1,\nu_2,\nu_3) \tag{5.2}$$

Applying the approach proposed above, we obtain the necessary condition for the existence of an IR (5.1) for Eqs (1.1) and (1.2):

$$\left[(\overline{\omega}_3 v_2 - \overline{\omega}_2 v_3) - (\overline{\omega}_1 v_3 - \overline{\omega}_3 v_1)\frac{\partial\psi}{\partial x_2} - (\overline{\omega}_2 v_1 - \overline{\omega}_1 v_2)\frac{\partial\psi}{\partial x_3}\right]\times \tag{5.3}$$

$$\times\left[\mu(v_1, v_2, v_3) - \left(\frac{\partial\psi}{\partial v_1} + \frac{\partial\psi}{\partial x_2}\frac{\partial\theta_2}{\partial v_1} + \frac{\partial\psi}{\partial x_3}\frac{\partial\theta_3}{\partial v_1} + \frac{\partial\theta_2}{\partial v_2} + \frac{\partial\theta_3}{\partial v_3}\right)\right] = 0$$

where, because $a_{ij} = a_{ji}$ $(i \neq j)$,

$$\overline{\omega}_i = a_{i1}\theta_1(v_1, v_2, v_3) + a_{i2}\theta_2(v_1, v_2, v_3) + a_{i3}\theta_3(v_1, v_2, v_3), \quad i = 1, 2, 3$$

Disregarding the singular case in which the first bracketed expression in (5.3) vanishes (the analogue of the case $\Delta = 0$ considered in Section 3), we deduce from (5.3), by virtue of (5.2), that

$$\mu(v_1, v_2, v_3) = \frac{\partial\theta_1}{\partial v_1} + \frac{\partial\theta_2}{\partial v_2} + \frac{\partial\theta_3}{\partial v_3} \tag{5.4}$$

Thus, if Eq. (5.4) holds, Eqs (1.1) and (1.2) admit of an IR (5.1) of layer zero using first integrals. In practice, it is natural to convert (5.4) to a form in which function (5.1), its derivatives, and also the given functions $L(v_1, v_2, v_3)$ and $U(v_1, v_2, v_3)$ occur explicitly.

6. CONCLUSION

The results established above have been obtained for IRs of layer zero. The fact that they do not carry over directly to the case of IRs of other layers will be considered by an example in which we investigate an IR of layer one without the use of first integrals, at the stage of evaluating the derivatives of the IR. Suppose that in Eqs (1.1) and (1.2)

$$\mu(v_1, v_2, v_3) \equiv 0, \quad L = \lambda \cdot v - \frac{1}{2}(Bv \cdot v), \quad U = s \cdot v - \frac{1}{2}(Cv \cdot v)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $s = (s_1, s_2, s_3)$ are vectors with constant components and B and C are constant symmetric 3×3 matrices. Consider an IR of these equations in the form [8]

$$f = x_1 - b_1 v_1 - b_3 v_3 = 0 \tag{6.1}$$

(b_1 and b_2 are constants). The parameters of Eqs (1.1) and (1.2) must obey the conditions

$$a_{ij} = 0 \ (i \neq j), \quad a_{ii} = a_i, \quad \lambda_1 = \lambda_2 = 0, \quad B_{ij} = 0 \ (i \neq j)$$

$$s_1 = a_1\lambda_3 b_3, \quad s_2 = a_2\lambda_3 c_3, \quad s_3 = -a_1 b_1 \lambda_3$$

$$B_{11} = (a_2 a_3 - a_1 a_2 - a_1 a_3) b_1 a_2^{-1} a_3^{-1}, \quad B_{22} = (a_1 a_3 - a_1 a_2 - a_2 a_3) b_1 a_2^{-1} a_3^{-1}$$

$$B_{33} = (a_1 a_2 - a_1 a_3 - a_2 a_3) b_1 a_2^{-1} a_3^{-1}$$

$$C_{12} = -a_1 a_2 b_3 c_3 a_3^{-1}, \quad C_{13} = (a_1 - a_3) a_1 b_1 b_3 a_3^{-1}$$

$$C_{23} = (a_2 - a_3) a_1 b_1 c_3 a_3^{-1}$$

$$C_{11} = C_{33} + a_2 c_3^2 + a_1(a_3 - a_1)(b_3^2 - b_1^2) a_3^{-1}$$

$$C_{22} = C_{33} + a_1 b_3^2 + (a_3 - a_2)(a_2^2 c_3^2 - a_1^2 b_1^2) a_2^{-1} a_3^{-1}$$

where c_3 is an arbitrary constant. Let us evaluate the derivatives of the function on the left of (6.1) along trajectories of Eqs (1.1) and (1.2), on the assumption that the right-hand sides of the latter satisfy the above conditions, and then substitute the value of x_1 from (6.1) into the resulting equations. We obtain

$$\frac{df}{dt}=\varphi(x_2,\nu_2,\nu_3)\sigma(x_3,\nu_1,\nu_2,\nu_3),\quad \frac{d^2f}{dt^2}=\varphi(x_2,\nu_2,\nu_3)\frac{d\sigma}{dt}(x_3,\nu_1,\nu_2,\nu_3) \tag{6.2}$$

where

$$\varphi(x_2,\nu_2,\nu_3)=x_2-\frac{a_1b_1}{a_2}\nu_2-c_3\nu_3$$

$$\sigma(x_3,\nu_1,\nu_2,\nu_3)=(a_3-a_2)x_3-a_2\lambda_3-\frac{a_1a_2b_3}{a_3}\nu_1+\frac{a_3-a_2}{a_3}(a_2c_3\nu_2-a_1b_3\nu_3)$$

Expressions (6.2) vanish provided that

$$\varphi(x_2,\nu_2,\nu_3)=0 \tag{6.3}$$

Equations (1.1) and (1.2) on the IRs (6.1) and (6.3) reduce to the form

$$\begin{aligned}
a_3\dot{x}_3&=(a_2c_3\nu_1-a_1b_3\nu_2)[a_1b_1\nu_3-a_3\lambda_3-(a_1b_3\nu_1+a_2c_3\nu_2)]\\
\dot{\nu}_1&=a_3x_3\nu_2-(a_1b_1\nu_2+a_3c_3\nu_3)\nu_3\\
\dot{\nu}_2&=-a_3x_3\nu_1+a_1(b_1\nu_1+b_3\nu_3)\nu_3\\
\dot{\nu}_3&=(a_2c_3\nu_1-a_1b_3\nu_2)\nu_3
\end{aligned} \tag{6.4}$$

and they have first integrals

$$\begin{aligned}
&\nu_1^2+\nu_2^2+\nu_3^2=1\\
&x_3=\frac{1}{a_3}(l_0\nu_3^{-1}+d_0\nu_3)\\
&(a_1b_3\nu_1+a_2c_3\nu_2)\nu_3+a_3\lambda_3\nu_3+(d_0-a_1b_1)\nu_3^2=l_0
\end{aligned} \tag{6.5}$$

where l_0 and d_0 are arbitrary constants. Hence the problem of integrating Eqs (6.4) reduces to quadratures. Using (6.1), (6.3) and (6.5), we find x_1, x_2, x_3 as functions of ν_1, ν_2, ν_3:

$$x_1=b_1\nu_1+b_3\nu_3,\quad x_2=\frac{1}{2}(a_1b_1\nu_2+a_2c_3\nu_3),\quad x_3=\frac{1}{a_3}(l_0\nu_3^{-1}+d_0\nu_3)$$

for which conditions (3.10) do not hold.

Equation (6.1) is an IR of layer one in P. V. Kharlamov's terminology [2], but not of layer zero, as in Sections 2–5 of this paper. Thus, IRs of layer one with $\mu\equiv 0$ require further investigation.

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